DIFFERENTIAL EQUATIONS WITH A CONTINUOUS INFINITUDE OF VARIABLES.

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The existence of solutions of the simultaneous system of differential equations

$$\frac{du_\rho}{d\tau} = f_\rho(\tau, u_1, \cdots u_\nu), \qquad (\rho = 1 \cdots \nu)$$

has been proved by means of various devices. Moreover the nature of the solutions when regarded as functions of the initial constants has also been treated.*

Some of these results have been extended to differential equations with a denumerable infinitude of variables by Von Koch, Moulton and Hart.†

In 1911, G. Kowalewski discussed the existence theorem for differential equations which involved Schmidt integral power series,‡ and in 1914, Volterra stated an existence theorem for general differential equations involving functions of lines.§ He gives, however, few details of the proof.

It is intended in the first section of this paper to restate this theorem of Volterra, and to give the details of the proof. The problem may be stated somewhat more explicitly as follows. Given the equation

$$\frac{\partial u(\xi, \tau)}{\partial \tau} = f[\xi, \tau, u(\xi')], \tag{1}$$

where ξ, ξ' are real variables on the range (0, 1), τ is a real variable on $|\tau - \tau_0| \leqq \alpha$, $u(\xi')$ has the range of continuous functions for which $\max |u(\xi') - u_0(\xi')| \leqq \beta$, and $f[\xi, \tau, u]$ is a functional eliminating the argument ξ' and yielding for each ξ, τ, u of the above ranges a real number.

* See, for example, the two articles by Bliss: "The Solutions of Differential Equations of the First Order as Functions of the Initial Values," *Annals of Mathematics*, 2d Series, Vol. 6 (1905), p. 49; "Solutions of Differential Equations as Functions of the Constants of Integration," *Bulletin of the American Mathematical Society*, Vol. XXIV (1918), p. 15. For other references, see Encyclopädie der Mathematischen Wissenschaften, II, A4a, p. 195, 200.

† H. Von Koch, "Ofversigt af Konliga Vetenskaps Akademiens Fordhandligar," Vol. 56 (1899), pp. 395–411.

F. R. Moulton, *Proceedings of the National Academy of Sciences*, Vol. 1, pp. 350–354.

W. L. Hart, "Differential Equations and Implicit Functions in Infinitely Many Variables, "*Transactions of the American Mathematical Society*, Vol. 18 (1917), pp. 125–160.

‡ K. Kowalewski, "Ueber Funktionraume II," *Wiener Berichte*, Vol. 120 (1911), ab. 2a.

§ Volterra, "Equazioni integro-differenziali ed equazioni alle derivate funzionali," *Rendiconti della reale Accademia dei Lincei*, 23 serie V (1914), p. 55.

The problem is to find hypotheses which are sufficient to insure the existence of a unique solution $u(\xi, \tau)$ reducing for an arbitrary value $\bar{\tau}$ of the above range for τ to an arbitrary continuous function $\bar{u}(\xi)$ of the above range for u.

After proving the existence of a solution through a fixed initial element, one is naturally interested in knowing the character of the solutions when the latter are considered as depending not only on the variables ξ, τ but also on the initial elements. It is shown in section 3 that with suitable hypotheses, the solutions are continuous functionals of the initial elements. In order to proceed farther with these questions, it was found convenient to use the concept of "difference function" due to Bliss which is a modification of Fréchet's differential. Four lemmas which are found to be useful in the sequel are proved in section 2. In section 4 a theorem is given which tells under what circumstances the solutions will possess difference functions with regard to the initial elements. This is an analogue of a corresponding theorem in differential equations with a finite number of variables, viz., the theorem concerning differentiability with respect to the initial constants. In the last section equations involving integral power series are considered and the connection with Kowalewski's paper is brought out. In a subsequent paper an application of these results will be made in proving that there exists a solution of an equation with a continuous infinitude of variables analogous to a linear partial differential equation in a finite number of variables.

§1. EXISTENCE OF A SOLUTION THROUGH GIVEN INITIAL ELEMENTS.

Let τ and ξ denote real variables and $u(\xi')$ a real-valued continuous function on the interval $0 \leqq \xi \leqq 1$. Moreover let A be a set of elements $[\xi, \tau, u(\xi')]$ defined by the conditions

$$(A) \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \alpha, \qquad ||u(\xi') - u_0(\xi')|| \leqq \beta,$$

where the notation $||\quad||$ means that the maximum is taken with respect to all ξ''s of the interval (0, 1) and where τ_0, u_0 stand for a particular real number and a particular real continuous function of the above ranges respectively. Finally, let $f[\xi, \tau, u(\xi')]$ be a real single-valued functional eliminating the argument ξ'. With these notations in mind, one may state and prove the following existence theorem.

THEOREM 1. *Let the functional f satisfy the hypotheses*

(H_1) *For each function u for which (ξ, τ, u) belongs to (A), $f[\xi, \tau, u]$ is continuous in $0 \leqq \xi \leqq 1$, $|\tau - \tau_0| \leqq \alpha$.*

(H_2) *There exists a number κ such that for every pair of elements (ξ, τ, u'), (ξ, τ, u'') of the set (A), the inequality*

$$|f[\xi, \tau, u'] - f[\xi, \tau, u'']| \leqq \kappa ||u' - u''||$$

holds.

Under these circumstances, $|f|$ *has an upper bound* γ *in* (A), *and there exists one and but one function* $v(\xi, \tau)$ *such that*

(C_1) $v(\xi, \tau)$, $v_\tau(\xi, \tau)$ *are continuous and the element* $[\xi, \tau, v(\xi, \tau')]$ *is in* (A) *provided that* (ξ, τ) *is in the following set* B:

$$(B) \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \rho,$$

where ρ *is the smaller of* α *and* β/γ, *and* γ *is an upper bound of* $|f[\xi, \tau, u]|$ *in the set* (A).

$$(C_2) \qquad v(\xi, \tau_0) = u_0(\xi).$$

$$(C_3) \qquad v_\tau(\xi, \tau) = f[\xi, \tau, v(\xi', \tau)].$$

It will be found helpful both in simplifying the proof and as a means for future reference to consider the following lemmas.

LEMMA 1. *If the functional* f *satisfies the hypotheses of theorem* 1, *then it is bounded, i.e., there exists a number* γ *independent of whatever element* (ξ, τ, u) *of the set* (A), *is considered for which* $|f[\xi, \tau, u]| \leqq \gamma$.

For, one may write

$$|f[\xi, \tau, u]| \leqq |f[\xi, \tau, u] - f[\xi, \tau, u_0]| + |f[\xi, \tau, u_0]|,$$

where the elements (ξ, τ, u) and (ξ, τ, u_0) are taken in the set (A). By (H_2) it is clear that the first expression on the right-hand side is less than $\kappa\beta$ and by (H_1) the last term is at most μ where $\mu = \| f[\xi, \tau, u_0] \|$.

LEMMA 2. *If* $u(\xi, \tau)$ *is continuous in* (B) *and the element* $[\xi, \tau, u(\xi', \tau)]$ *is in* (A) *for every* (ξ, τ) *of* (B), *then* $f[\xi, \tau, u]$ *is continuous in* (B).

Since the element (ξ, τ, u) is in (A), one has by (H_2) the following inequality.

$$\begin{aligned}
&|f[\xi + \Delta\xi, \tau + \Delta\tau, u[\xi', \tau + \Delta\tau)] - f[\xi, \tau, u(\xi', \tau)]| \\
&\qquad \leqq |f[\xi + \Delta\xi, \tau + \Delta\tau, u(\xi', \tau + \Delta\tau)] - f[\xi + \Delta\xi, \tau + \Delta\tau, u(\xi', \tau)]| \\
&\qquad\quad + |f[\xi + \Delta\xi, \tau + \Delta\tau, u(\xi', \tau)] - f[\xi, \tau, u(\xi', \tau)]| \\
&\qquad \leqq \kappa \| u(\xi', \tau + \Delta\tau) - u(\xi', \tau) \| \\
&\qquad\quad + |f[\xi + \Delta\xi, \tau + \Delta\tau, u(\xi', \tau)] - f[\xi, \tau, u(\xi', \tau)]|.
\end{aligned}$$

But, since $u(\xi, \tau)$ is continuous in (B), $\| u(\xi', \tau + \Delta\tau) - u(\xi', \tau) \|$ can be made as small as desired provided $|\Delta\tau|$ is taken small enough. Also, the expression $|f[\xi + \Delta\xi, \tau + \Delta\tau, u(\xi', \tau)] - f[\xi, \tau, u(\xi', \tau)]|$ can be made arbitrarily small by restricting sufficiently $|\Delta\xi|$, $|\Delta\tau|$, as follows from (H_1). Hence, the truth of the lemma follows.

Now, let $\{v_\nu(\xi, \tau)\}$ be a sequence of functions defined by the relations

$$(2) \qquad \begin{aligned} v_0(\xi, \tau) &= u_0(\xi), \\ v_{\nu+1}(\xi, \tau) &= u_0(\xi) + \int_{\tau_0}^{\tau} f[\xi, \tau', v_\nu(\xi', \tau')]d\tau', \qquad (\nu = 0, 1, \cdots). \end{aligned}$$

LEMMA 3.

(1) *Each* $v_\nu(\xi, \tau)$ *is continuous in* (B).

(2) *If* (ξ, τ) *is in* (B), *then for each* ν *the element* $[\xi, \tau, v_\nu(\xi', \tau)]$ *is in* (A).

(3) *For each* ν *the functional* $f[\xi, \tau, v_\nu(\xi', \tau)]$ *is continuous in* (B).

By (H_1) the conclusions of the lemma including (3) are true for $\nu = 0$. Moreover since by Lemma 1, one has the inequality

$$|v_1(\xi, \tau) - u_0(\xi)| \leqq |\textstyle\int_{\tau_0}^{\tau} f[\xi, \tau', u_0(\xi')]d\tau'| \leqq \gamma |\tau - \tau_0|,$$

the conclusion (2) follows for $\nu = 1$ and consequently (3) for $\nu = 1$ by Lemma 2. Thus $v_2(\xi, \tau)$ is continuous in (B). By sequentially using

$$|v_{\nu+1}(\xi, \tau) - u_0(\xi)| \leqq \gamma |\tau - \tau_0|$$

and Lemma 2, one easily obtains the results of the Lemma.

LEMMA 4. *The sequence* $\{v_\nu(\xi, \tau)\}$ *converges uniformly in* (B) *to a function* $v(\xi, \tau)$ *continuous in* (B), *and such that* (ξ, τ, v) *is in* (A) *whenever* (ξ, τ) *is in* (B).

From the defining equations (2) one has

$$|v_1(\xi, \tau) - v_0(\xi, \tau)| \leqq \mu |\tau - \tau_0|,$$

where $\mu = \| f[\xi, \tau, u_0] \|$ in (A). To complete the induction, assume

$$|v_\nu(\xi, \tau) - v_{\nu-1}(\xi, \tau)| \leqq \frac{\mu}{\kappa} \frac{\kappa^\nu |\tau - \tau_0|^\nu}{\nu!}.$$

It follows from this and hypothesis (H_2) that

$$|v_{\nu+1}(\xi, \tau) - v_\nu(\xi, \tau)| = |\textstyle\int_{\tau_0}^{\tau} \{f[\xi, \tau', v_\nu(\xi', \tau')] - f[\xi, \tau', v_{\nu-1}(\xi', \tau')]\} d\tau'|$$

$$\leqq \left| \int_{\tau_0}^{\tau} \frac{\mu}{\kappa} \frac{\kappa^{\nu+1}}{\nu!} |\tau' - \tau_0|^\nu d\tau' \right| \leqq \frac{\mu}{\kappa} \frac{\kappa^{\nu+1} |\tau - \tau_0|^{\nu+1}}{(\nu + 1)!}.$$

Now, consider the series

(3) $v(\xi, \tau) = u_0(\xi) + [v_1(\xi, \tau) - u_0(\xi)] + [v_2(\xi, \tau) - v_1(\xi, \tau)] + \cdots.$

The absolute value of every term of (3) is less than the corresponding term of the expansion of $(e^{\kappa|\tau - \tau_0|} - 1)\mu/\kappa$. Since this dominating series converges surely for all elements (ξ, τ) of (B), it follows that the series (3) converges absolutely uniformly in (B) and hence uniformly for all (ξ, τ) of the same set. Moreover, since each term of (3) is continuous in (B) it follows that $v(\xi, \tau)$ is continuous in (B) and hence by Lemma 2, the last conclusion of Lemma 4 follows.

LEMMA 5. *The function* $v(\xi, \tau)$ *satisfies the relation*

$$v(\xi, \tau) = u_0(\xi) + \textstyle\int_{\tau_0}^{\tau} f[\xi, \tau', v(\xi', \tau')]d\tau', \qquad \text{for } \nu \geqq \nu_0.$$

For, by Lemma 4 there exists a sufficiently large index ν_0 such that the maximum with respect to both (ξ, τ) of (B) of the expression $|v_\nu(\xi, \tau) - v(\xi, \tau)|$

can be made as small as desired, say less than ϵ. Hence, since the elements (ξ, τ, v_ν) and (ξ, τ, v) are in (A), it follows by (H_2) of Theorem 1 that

$$|\int_{\tau_0}^{\tau}\{f[\xi, \tau', v_\nu(\xi', \tau')] - f[\xi, \tau', v(\xi', \tau')]\}d\tau' \leqq \kappa\epsilon|\tau - \tau_0|$$

or

$$\lim_{\nu \to \infty} \int_{\tau_0}^{\tau} f[\xi, \tau', v_\nu(\xi', \tau')]d\tau' = \int_{\tau_0}^{\tau} f[\xi, \tau', v(\xi', \tau')]d\tau'.$$

But, by Lemma 4, $v(\xi, \tau) = \lim_{\nu \to \infty} v_\nu(\xi, \tau)$ so that it follows from the relations (2) that

$$v(\xi, \tau) = u_0(\xi) + \int_{\tau_0}^{\tau} f[\xi, \tau', v(\xi', \tau')]d\tau!$$

as desired.

On differentiating the last relation with respect to τ, one sees that $v_\tau(\xi, \tau)$ is continuous in (B) and satisfies the differential equation (1). Moreover by putting $\tau = \tau_0$ in this same equation, one finds that the solution reduces to the specified initial function $u_0(\xi)$.

It remains only to prove the uniqueness of the solution.

LEMMA 6. *The equation* (1) *has only one solution* $v(\xi, \tau)$ *which is continuous in* (B), *for which* $v_\tau(\xi, \tau)$ *is continuous in* (B) *and which reduces to* $u_0(\xi)$ for $\tau = \tau_0$.

Suppose that besides $v(\xi, \tau)$ there could be another solution $\bar{v}(\xi, \tau)$ having the properties stated in the lemma. Then this solution also would have to satisfy a relation like that in Lemma 5. It would then follow by (H_2) of Theorem 1, since both (ξ, τ, v) and $(\xi, \tau, \bar{v})$ are in (A), that

$$(4) \quad |v(\xi, \tau) - \bar{v}(\xi, \tau)| = |\int_{\tau_0}^{\tau} \{f[\xi, \tau', v(\xi', \tau')] - f[\xi, \tau', \bar{v}(\xi', \tau')]\}d\tau'| \leqq \tau\kappa\beta|\tau - \tau_0|.$$

If now (H_2) is sequentially applied to the second member of (4) and use is made of the last inequality, one obtains finally the inequality

$$|v(\xi, \tau) - \bar{v}(\xi, \tau)| \leqq \frac{2\beta\kappa^\nu|\tau - \tau_0|^\nu}{\nu!}$$

from which follows the statement of the Lemma by a passage to the limit.

This completes the proof of Theorem I.

The following theorem, which is really a Corollary of Theorem I, will be found useful.

THEOREM II. *Let a set* (A') *be defined by*

$$(A') \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \alpha, \qquad ||u - u_0|| \leqq \infty.$$

If the hypotheses of Theorem I are satisfied in this set and if $||f[\xi, \tau, u_0]||$, *then the conclusions of Theorem I will hold for the set of points* (B'),

$$(B') \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \alpha.$$

Part (2) of Lemma 3 which deals with proving that the element (ξ, τ, v) is in (A') is of course unnecessary here. The smaller of α and β/γ should in this case be replaced evidently by the number α. The convergence and uniqueness proofs go through precisely as before.

§ 2. SOME DEFINITIONS AND LEMMAS CONCERNING FUNCTIONALS AND DIFFERENCE FUNCTIONS.

Before proceeding to a discussion of other properties of the solutions of equation (I), it will be convenient to give a few definitions and lemmas to which frequent reference will be made in the sequel.

Let η stand for the set of elements $(\eta_1 \cdots \eta_r)$ where the $\eta_1 \cdots \eta_r$ are real variables with ranges which are composed of continuous intervals or discrete sets. For example, η_1 might have the range $1, 1/2, 1/3, \cdots$ and all the other η's vary over $(0, 1)$. Moreover, let u stand for the set of elements $(u_1 \cdots u_s)$ where $u_1 \cdots u_s$ are real continuous functions of ξ' on the interval $0 \leqq \xi' \leqq 1$. Consider now a real-valued functional operation $v[\xi, \eta, u] = v[\xi, \eta_1 \cdots \eta_r, u_1 \cdots u_s)$ defined by the relations

$$(R) \qquad \begin{aligned} 0 \leqq \xi \leqq 1, \qquad |\eta_i - \eta_i^{(0)}| \leqq \lambda_i \qquad (i = 1 \cdots r), \\ ||u_i - u_i^{(0)}|| \leqq \mu_i \qquad (i = 1 \cdots s). \end{aligned}$$

In other words, $v[\xi, \eta, u]$ is of such a character that whenever the element (ξ, η, u) of the set (R) is given, the correspondence v determines a real number. Unless otherwise specified, attention will be confined to η's which represent sets of variables having continuous ranges.

Definition.—*$v[\xi, \eta, u]$ is said to be continuous in the set (R) if for every positive number ϵ and for every u for which (ξ, η, u) is in (R), there exists another positive number $\delta_{\epsilon,u}$ which is independent of (ξ, η) such that the inequality $|v[\bar{\xi}, \bar{\eta}, \bar{u}] - v[\xi, \eta, u]| \leqq \epsilon$ holds if (ξ, η, u) and $(\xi, \bar{\eta}, \bar{u})$ are in (R) and if $|\bar{\xi} - \xi| \leqq \delta_{\epsilon,u}$, $|\bar{\eta} - \eta| \leqq \delta_{\epsilon,u}$, $||\bar{u} - u|| \leqq \delta_{\epsilon,u}$.*

The notation $|\bar{\eta} - \eta|$ means of course $|\bar{\eta}_i - \eta_i|$, $(i = 1 \cdots r)$ and $||\bar{u} - u||$ stands for $||\bar{u}_i - u_i||$, $(i = 1 \cdots s)$.

LEMMA 1. *If*

(1) *$v_\nu[\xi, \eta, u]$ is continuous in (R),*

(2) *The sequence $\{v_\nu[\xi, \eta, u]\}$ tends to a limit $v[\xi, \eta, u]$ uniformly with respect to all elements $[\xi, \eta, u]$ of (R), then, $v[\xi, \eta, u]$ is continuous in (R).*

In the inequality

$$\begin{aligned} |v[\bar{\xi}, \bar{\eta}, \bar{u}] - v[\xi, \eta, u]| &\leqq |v[\bar{\xi}, \bar{\eta}, \bar{u}] - v_{\nu_0}[\bar{\xi}, \bar{\eta}, \bar{u}]| \\ &+ |v_{\nu_0}[\bar{\xi}, \bar{\eta}, \bar{u}] - v_{\nu_0}[\xi, \eta, u]| + |v_{\nu_0}[\xi, \eta, u] - v[\xi, \eta, u]| \end{aligned}$$

one can choose ν_0 so large that

$$|v[\bar{\xi}, \bar{\eta}, \bar{u}] - v_{\nu_0}[\bar{\xi}, \bar{\eta}, \bar{u}]| + |v_{\nu_0}[\xi, \eta, u] - v[\xi, \eta, u]| \leqq \frac{2\epsilon}{3}$$

by the uniformity of the convergence. Moreover by the continuity of $v_{\nu_0}[\xi, \eta, u]$ there exists a number $\delta_{\epsilon, u, \nu_0}$ for which

$$|v_{\nu_0}[\bar{\xi}, \bar{\eta}, \bar{u}] - v_{\nu_0}[\xi, \eta, u]| \leqq \epsilon/3.$$

Take this $\delta_{\epsilon, u, \nu_0}$ as the δ associated with the continuity of $v[\xi, \eta, u]$.

Let $(\tilde{R})$ be a set of elements $[\tilde{\eta}, \tilde{u}]$ where $\tilde{\eta}$ stands for a set $(\tilde{\eta}_1 \cdots \tilde{\eta}_r)$ with each $\tilde{\eta}_i$ ranging over the whole linear continuum and $\tilde{u}$ stands for the set of functions $(\tilde{u}_1 \cdots \tilde{u}_s)$ where the $\tilde{u}_i$ are arbitrary continuous functions of ξ' on the interval $(0, 1)$. In all that follows in this section the elements (ξ, η, u) and $(\bar{\xi}, \bar{\eta}, \bar{u})$ shall be understood as belonging to (R) and $(\tilde{\eta}, \tilde{u})$ as belonging to $(\tilde{R})$.

Definition of the function $a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$.

(1) *a is continuous in $[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ in the sense defined above.*

(2) $a[\xi, \eta, u, \bar{\eta}, \bar{u}; \gamma_1\tilde{\eta}_1 + \gamma_2\tilde{\eta}_2, \gamma_1\tilde{u}_1 + \gamma_2\tilde{u}_2] = \gamma_1 a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_1, \tilde{u}_1] + \gamma_2 a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_2, \tilde{u}_2]$, *where $(\tilde{\eta}_1, \tilde{u}_1)$, $(\tilde{\eta}_2, \tilde{u}_2)$ are elements of $(\tilde{R})$ and γ_1, γ_2 are arbitrary real numbers.*

(3) *There exists a positive number μ independent of $[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ for which*

$$|a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]| \leqq \mu||\tilde{\eta}, \tilde{u}||.$$

The notation $||\tilde{\eta}, \tilde{u}||$ means the larger of $[||\tilde{\eta}||, ||\tilde{u}||]$.

Definition of a difference function.

$a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ is said to be a difference function of $f[\xi, \eta, u]$ if a has the properties (1), (2), (3) *just defined and satisfies the relation*

$$f[\xi, \bar{\eta}, \bar{u}] - f[\xi, \eta, u] = a[\xi, \eta, u, \bar{\eta}, \bar{u}; \bar{\eta} - \eta, \bar{u} - u].$$

Besides the properties (1), (2), (3) of a there will be occasion to consider a fourth property.

(1′) *If in a, one substitutes for u, $\bar{u}$, $\tilde{u}$ the functionals $v'[\xi, \eta', u']$, $v''[\xi, \eta'', u'']$, $v'''[\xi, \eta''', u''']$ respectively, where v', v'', v''' are continuous in the sense above defined, then the resulting expression considered as depending on $\xi, \eta, \bar{\eta}, \tilde{\eta}, \eta', \eta'', \eta''', u', u'', u'''$ is also continuous in the same sense.*

It should be understood in (1′) that η', η'', η''' may denote sets of variables which have both discrete and continuous ranges. If for example η_1 has the range $1, 1/2, 1/3 \cdots$, then $v(\eta_1)$ is continuous at the point 0 if $|v(\eta_1') - v(0)| \leqq \epsilon$ whenever $|\eta_1'| \leqq \delta$ or what amounts to the same thing $\lim v(\eta_1) = v(0)$ when $\lim \eta_1 = 0$. The u', u'', u''' must have ranges given by (R). This last property is denoted by (1′) because it is similar to (1) but much stronger. It will be understood therefore that whenever (1′) is used it will replace (1).

The following lemmas will be found very useful.

LEMMA 2. *If $f[\xi, \eta, u]$ has a difference function $a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ in*

the set (R), *it satisfies the Lipschitz condition described in hypothesis* (H_2) *of Theorem I in this set.*

For, taking $\bar{\eta} = \eta$ and u, $\bar{u}$ as single real continuous functions, one sees from the definition of a difference function and property (3) that

$$|f[\xi, \eta, \bar{u}] - f[\xi, \eta, u]| \leqq \mu||\bar{u} - u||.$$

LEMMA 3. *If* $f[\xi, \eta, u]$ *has a difference function in* (R), *it is continuous in* (R) *in the sense described above.*

For, if in the inequality

$$|f[\bar{\xi}, \bar{\eta}, \bar{u}] - f[\xi, \eta, u]| \leqq |f[\bar{\xi}, \bar{\eta}, \bar{u}] - f[\bar{\xi}, \eta, u]| + |f[\bar{\xi}, \eta, u] - f[\xi, \eta, u]|$$

one takes $||\bar{\eta} - \eta||$, $||\bar{u} - u||$ sufficiently small, one can make $|f[\xi, \bar{\eta}, \bar{u}] - f[\xi, \eta, u]|$ as small as desired, since it is less than $\mu||\bar{\eta} - \eta, \bar{u} - u||$. Furthermore by property (1) of the difference function the second expression on the right tends to zero with $|\bar{\xi} - \xi|$. It is to be noted here that because of the character of the μ the continuity is surely of the type defined in the beginning of this section and even stronger.

LEMMA 4. *Let* $v_\nu[\xi, \eta, u]$ *be a sequence of functionals with the properties:*

(1) *The sequence converges to a limit function* $v[\xi, \eta, u]$.

(2) *Each* $v_\nu[\xi, \eta, u]$ *has a difference function* $b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ *with all of which are associated the same constant* μ.

(3) *The sequence* $\{b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]\}$ *converges to a limit* $b[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ *uniformly with respect to all* (ξ, η, u) *in* (R), $(\bar{\xi}, \bar{\eta}, \bar{u})$ *in* (R) *and* $(\tilde{\eta}, \tilde{u})$ *for which* $||\tilde{\eta}, \tilde{u}|| \leqq 1$.

Then, the limit $b[\xi, \eta, u, \eta, \bar{u}; \tilde{\eta}, \tilde{u}]$ *has the properties* (1), (2) *and* (3) *and is a difference function for* $v[\xi, \eta, u]$.

From (2) one has the relation

$$v_\nu[\xi, \bar{\eta}, \bar{u}] - v_\nu[\xi, \eta, u] = b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \eta - \eta, \bar{u} - u], \qquad \nu = 1, 2, \cdots,$$

so that passing to the limit as is allowable by (1) and (3), it is clear that

$$v[\xi, \bar{\eta}, \bar{u}] - v[\xi, \eta, u] = b[\xi, \eta, u, \bar{\eta}, \bar{u}, \bar{\eta} - \eta, \bar{u} - u].$$

It remains to show that $b[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ has the properties (1), (2), (3) of a difference function. By (2) and (3) one sees that

$$\begin{aligned} b[\xi, \eta, u, \bar{\eta}, \bar{u}; \gamma_1\tilde{\eta}_1 + \gamma_2\tilde{\eta}_2, \gamma_1\tilde{u}_1 + \gamma_2\tilde{u}_2] \\ &= \lim_{\nu\to\infty} b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \gamma_1\tilde{\eta}_1 + \gamma_2\tilde{\eta}_2, \gamma_1\tilde{u}_1 + \gamma_2\tilde{u}_2] \\ &= \lim_{\nu\to\infty} \{\gamma_1 b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_1, \tilde{u}_1] + \gamma_2 b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_2, \tilde{u}_2]. \end{aligned}$$

Since each term on the right-hand side separately converges, it follows that

$$\begin{aligned} b[\xi, \eta, u, \eta, \bar{u}; \gamma_1\tilde{\eta}_1 + \gamma_2\tilde{\eta}_2, \gamma_1\tilde{u}_1 + \gamma_2\tilde{u}_2] \\ &= \gamma_1 b[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_1, \tilde{u}_1] + \gamma_2 b[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_2, \tilde{u}_2] \end{aligned}$$

which proves property (2). Moreover since each b_ν is continuous in ξ, η, u, $\bar{\eta}$, $\bar{u}$, $\tilde{\eta}$, $\tilde{u}$ and since the sequence $\{b_\nu\}$ converges uniformly to b, it follows from Lemma I that the limit functional b will also be continuous in the same arguments. Finally, property (3) is an immediate consequence of hypothesis (2).

§ 3. CONTINUITY OF THE SOLUTIONS WITH RESPECT TO THE INITIAL ELEMENTS.

Consider now a set (A_0) of elements (ξ, τ, u) defined by the inequalities

$$(A_0) \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_{00}| \leqq \alpha + \delta, \qquad \| u - u_{00} \| \leqq \beta + \delta,$$

where (ξ, τ_{00}, u_{00}) is a particular element. Assume that all the hypotheses of Theorem I are satisfied in this set. Consider also a set of elements (ξ, τ, τ_0, u_0) given by the relations,

$$(B_0) \quad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \rho, \qquad |\tau_0 - \tau_{00}| \leqq \delta, \qquad \| u_0 - u_{00} \| \leqq \delta,$$

where ρ as before is the smaller of α and β/γ.

The set (A) associated with every element (τ_0, u_0) for which $|\tau_0 - \tau_{00}| \leqq \delta$ and $\| u_0 - u_{00} \| \leqq \delta$ is a part of (A_0) as is clear from the inequalities

$$|\tau - \tau_{00}| \leqq |\tau - \tau_0| + |\tau_0 - \tau_{00}| \leqq \alpha + \delta,$$
$$\| u - u_{00} \| \leqq \| u - u_0 \| + \| u_0 - u_{00} \| \leqq \beta + \delta.$$

From this it follows that if one defines a sequence of approximating functionals $\{v_\nu[\xi, \tau, \tau_0, u_0]\}$ by equations (2) where now $[\xi, \tau, \tau_0, u_0]$ are all thought of as varying in the set (B_0), one can prove precisely as in Lemma 3, section 1, that these functionals are all defined in (B_0) and that the sequence converges uniformly in (B_0) to a solution $v[\xi, \tau, \tau_0, u_0]$ of the differential equation (1). One has thus secured a set of solutions which for all possible variations of the initial element $[\xi, \tau, \tau_0, u_0]$ in (B) has a common interval of definition $|\tau - \tau_0| \leqq \rho$.

THEOREM III. *If the hypotheses of Theorem I are satisfied in the set (A_0), then in the set (B_0) the solutions $v[\xi, \tau, \tau_0, u_0]$ are continuous functionals of their arguments in the sense of section 2.*

In the notations of section 2 the conclusion of this theorem has reference to a set $(R) = (B_0)$ with the elements $\xi = \xi$, $\eta = (\tau, \tau_0)$ and $u = u_0$. The proof of the theorem can be made to rest on the following

LEMMA. *If*

(1) *$f[\xi, \tau, u]$ satisfies the hypotheses of Theorem I in the set (A_0),*

(2) *$v[\xi, \tau, \tau_0, u_0]$ is continuous in (B_0),*

then, the functionals $g[\xi, \tau, \tau_0, u_0] = f[\xi, \tau, v[\xi', \tau, \tau_0, u_0]]$ and $\int_{\tau_0}^{\tau} g[\xi, \tau', \tau_0, u_0] d\tau'$ are also continuous in (B_0) in the same sense.

Consider any two elements $[\xi, \tau, \tau_0, u_0]$, $[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$ of (B_0). One may write

$$(5)\quad \begin{aligned} |g[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - g[\xi, \tau, \tau_0, u_0]| \leqq &|g[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - g[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, u_0]| \\ &+ |g[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, u_0] - g[\xi, \tau, \tau_0, u_0]|. \end{aligned}$$

Since f satisfies the Lipschitz condition in $(\bar{A}_0)$, it follows that the first expression on the right will not exceed $\kappa|v[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - v[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, u_0]|$. From the continuity of v it follows that this may be made as small as desired if $(\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0)$ is taken sufficiently near (ξ, τ, τ_0, u_0). It is to be remarked moreover that the associated δ is independent of ξ, τ, τ_0. Also, the second expression on the right-hand side of (5) can be made as small as one pleases, since it is equal to $|f[\bar{\xi}, \bar{\tau}, v[\xi', \bar{\tau}, \tau_0, u_0]] - f[\xi, \tau, v[\xi', \tau, \tau_0, u_0]]|$ and hence for each u_0 is uniformly continuous in ξ, τ, τ_0. This proves the first part of the Lemma.

Now, set

$$h[\xi, \tau, \tau_0, u_0] = \int_{\tau_0}^{\tau} g[\xi, \tau', \tau_0, u_0]d\tau'.$$

Then, it is clear that

$$\begin{aligned} |h[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - h[\xi, \tau, \tau_0, u_0]| \leqq &|\textstyle\int_{\tau}^{\bar{\tau}} g[\bar{\xi}, \tau', \bar{\tau}_0, \bar{u}_0]d\tau'| \\ &+ |\textstyle\int_{\bar{\tau}_0}^{\tau_0} g[\bar{\xi}, \tau', \bar{\tau}_0, \bar{u}_0]d\tau'| \\ &+ |\textstyle\int_{\tau_0}^{\tau} \{g[\bar{\xi}, \tau', \bar{\tau}_0, \bar{u}_0] - g[\xi, \tau', \tau_0, u_0]\}d\tau'|. \end{aligned}$$

Hence, applying the first part of the Lemma and Lemma 1 of section 1 to the first two terms on right, and the fact that $g[\xi, \tau, \tau_0, u_0]$ is continuous in (B_0), one sees that

$$|h[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - h[\xi, \tau, \tau_0, u_0]| \leqq \gamma(|\bar{\tau} - \tau| + |\bar{\tau}_0 - \tau_0| + \epsilon|\tau - \tau_0|)$$

if $(\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0)$ is sufficiently near to (ξ, τ, τ_0, u_0) which proves that $h[\xi, \tau, \tau_0, u_0]$ is continuous in (B_0). It is important to note that the uniformity of the continuity of $g[\xi, \tau, \tau_0, u_0]$ with respect to τ was needed in this discussion.

Consider now the approximating functionals

$$(2')\quad \begin{aligned} v_0[\xi, \tau, \tau_0, u_0] &= u_0(\xi), \\ v_{\nu+1}[\xi, \tau, \tau_0, u_0] &= u_0(\xi) + \textstyle\int_{\tau_0}^{\tau} f[\xi, \tau', v_\nu[\xi', \tau', \tau_0, u_0]]d\tau', \end{aligned}$$

$$\nu = 0, 1, 2, \cdots.$$

Suppose $[\xi, \tau, \tau_0, u_0]$ is any element of (B_0). Then for $\nu = 0$ one has

$$|\Delta v_0| = |v_0[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - v_0[\xi, \tau, \tau_0, u_0]| = |\bar{u}_0(\bar{\xi}) - u_0(\xi)|$$

so that

$$|\Delta v_0| \leqq |\bar{u}_0(\bar{\xi}) - u_0(\bar{\xi})| + |u_0(\bar{\xi}) - u_0(\xi)|.$$

But, if ϵ is arbitrarily assigned there exists a δ_{ϵ, u_0} independent of ξ such that if $|\bar{\xi} - \xi| \leqq \delta_{\epsilon, u_0}$ then $|u_0(\bar{\xi}) - u_0(\xi)| \leqq \epsilon$. This δ is independent of ξ

since u_0 is a continuous function on a closed interval. Also there exists a δ_ϵ such that if $\| \bar{u}_0 - u_0 \| \leqq \delta_\epsilon$ then $| \bar{u}_0(\bar{\xi}) - u_0(\bar{\xi}) | \leqq \epsilon$. This shows that $v_0[\xi, \tau, \tau_0, u_0]$ possesses precisely that type of continuity heretofore considered and may therefore be used as the $v[\xi, \tau, \tau_0, u_0]$ of the preceding Lemma. Thus by successive applications of this Lemma one gets the result that all the $v_\nu[\xi, \tau, \tau_0, u_0]$ are continuous in (B_0). Furthermore, as has already been remarked at the beginning of this section, the sequence $\{v_\nu[\xi, \tau, \tau_0, u_0]\}$ converges to a solution $v[\xi, \tau, \tau_0, u_0]$ uniformly in (B_0). Hence, using Lemma I of section 2 with $\eta = (\tau, \tau_0)$, $u = u_0$ one obtains the conclusion of Theorem III.

§ 4. DIFFERENTIABILITY OF THE SOLUTIONS WITH RESPECT TO THE INITIAL ELEMENTS.

The main result of this section is embodied in the following

THEOREM IV. *If*

(H_1) *The hypotheses of Theorem I are satisfied in* (A_0),

(H_2) $f[\xi, \tau, u]$ *has a difference function* $a[\xi, \tau, u, \bar{\tau}, \bar{u}; \tilde{\tau}, \tilde{u}]$ *in* (A_0),

$$f[\xi, \bar{\tau}, \bar{u}] - f[\xi, \tau, u] = a[\xi, \tau, u, \bar{\tau}, \bar{u}; \bar{\tau} - \tau, \bar{u} - u],$$

(H_3) *The difference function of* f *has the additional property* (1′) *of section* 2 *in* (A_0),

then, the solution $v[\xi, \tau, \tau_0, u_0]$ *has in* (B_0) *a difference function* $b[\xi, \tau, \tau_0, u_0 \bar{\tau}, \bar{\tau}_0 \bar{u}_0; \tilde{\tau} \tilde{\tau}_0 \tilde{u}_0]$.

The style of proof will be to interpret the sequence $\{v_\nu[\xi, \tau, \tau_0, u_0]\}$ with $\eta = (\tau, \tau_0)$ and $u = u_0$ of Lemma 4, section 2, as the sequence of approximating functionals (2′). If then one can show that all the hypotheses of that Lemma are satisfied here, Theorem IV will be proved.

LEMMA 1. *If*

(1) $f[\xi, \tau, u]$ *satisfies the hypotheses of Theorem IV*,

(2) $v[\xi, \tau, \tau_0, u_0]$ *has a difference function* $b[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0 \bar{u}_0; \tilde{\tau}, \tilde{\tau}_0, \tilde{u}_0]$ *in* (B_0),

(3) $(\xi, \tau, v[\xi', \tau, \tau_0, u_0])$ *is in* (A_0) *for every* (ξ, τ, τ_0, u_0) *of* (B_0),

then, $g[\xi, \tau, \tau_0, u_0] = f[\xi, \tau, v[\xi', \tau, \tau_0, u_0]]$ *and* $\int_{\tau_0}^{\tau} g[\xi, \tau', \tau_0, u_0] d\tau'$ *also have difference functions in* (B_0).

By hypothesis (1)

$$g[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - g[\xi, \tau, \tau_0, u_0] = a[\xi, \tau, v, \bar{\tau}, \bar{v}; \bar{\tau} - \tau, \bar{v} - v],$$

where $v = v[\xi, \tau, \tau_0, u_0]$ and $\bar{v} = v[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$. Moreover

$$v[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - v[\xi, \tau, \tau_0, u_0] \\ = b[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \bar{\tau} - \tau, \bar{\tau}_0 - \tau_0, \bar{u}_0 - u_0].$$

Hence,

$$g[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - g[\xi, \tau, \tau_0, u_0]$$
$$= c[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \bar{\tau} - \tau, \bar{\tau}_0 - \tau_0, \bar{u}_0 - u_0],$$

where

$$c[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0\bar{u}_0; \tilde{\tau}, \tilde{\tau}_0, \tilde{u}_0]$$
$$= a[\xi, \tau, v, \bar{\tau}, \bar{v}; \tilde{\tau}, b[\xi', \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \tilde{\tau}, \tilde{\tau}_0, \tilde{u}_0]].$$

It remains to show that c has properties (1), (2) and (3) of a difference function. Property (1) of c is an immediate consequence of the property (1′) possessed by the difference function a. Property (2) follows from property (2) of b and a. Since (ξ, τ, v), $(\xi, \bar{\tau}, \bar{v})$ are in (A_0), it is clear from property (3) of a that $|c| \leqq \mu \, \| \tilde{\tau}, b \|$ and hence from property (3) of b that $|c| \leqq \mu\lambda \, \| \tilde{\tau}, \tilde{\tau}_0, \tilde{u}_0 \|$, where λ is the constant associated with b. Thus property (3) of c is proved. This completes the proof that $g[\xi, \tau, \tau_0, u_0]$ has a difference function in (B).

Consider now the functional $h[\xi, \tau, \tau_0, u_0] = \int_{\tau_0}^{\tau} g[\xi, \tau', \tau_0, u_0]$. By means of an easy computation it is found that

$$\begin{aligned} k[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \tilde{\tau}, \tilde{\tau}_0, \tilde{u}_0] &= \tilde{\tau}\textstyle\int_0^1 g[\xi, \tau + \theta(\bar{\tau} - \tau), \bar{\tau}_0, \bar{u}_0]d\theta \\ &+ \tilde{\tau}_0\textstyle\int_1^0 g[\xi, \tau_0 + \theta(\bar{\tau}_0 - \tau_0), \bar{\tau}_0, \bar{u}_0]d\theta \\ &+ \textstyle\int_{\tau_0}^{\tau} c[\xi, \tau', \tau_0, u_0, \tau', \bar{\tau}_0, \bar{u}_0; 0, \tilde{\tau}_0, \tilde{u}_0]d\tau', \end{aligned} \tag{6}$$

where

$k[\xi, \tau, \tau_0, u_0\bar{\tau}\bar{\tau}_0\bar{u}_0; \bar{\tau} - \tau, \bar{\tau}_0 - \tau_0, \bar{u}_0 - u_0] = h[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - h[\xi, \tau, \tau_0, u_0]$. Properties (2) and (3) of k follow readily from the identity (6), properties (2) and (3) of g, the continuity of c, and elementary properties of integrals. The proof of property (1) follows readily from the fact that g and c are continuous in the sense of section 2.

LEMMA 2. *Every approximating functional $v_\nu[\xi, \tau, \tau_0, u_0]$ has a difference function b_ν in (B_0) if f satisfies the hypotheses of Theorem IV.*

It is clear from (2′) that $v_0[\xi, \tau, \tau_0, u_0]$ has a difference function $b_0 = \tilde{u}_0$. Then from Lemma 1, it will follow that every v_ν has a difference function b_ν where

$$b_\nu[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \bar{\tau} - \tau, \bar{\tau}_0 - \tau_0, \bar{u}_0 - u_0]$$
$$= v_\nu[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - v_\nu[\xi, \tau, \tau_0, u_0], \qquad \nu = 1, 2, \cdots.$$

It can be shown from the defining equations (2′) that

$$\begin{aligned} b_{\nu+1}[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \tilde{\tau}, \tilde{\tau}_0, \tilde{u}_0] &= \tilde{u}_0(\xi) \\ &+ \tilde{\tau}\textstyle\int_0^1 f[\xi, \tau + \theta(\bar{\tau} - \tau), v_\nu[\xi', \tau + \theta(\bar{\tau} - \tau), \bar{\tau}_0, \bar{u}_0]d\theta \\ &+ \tilde{\tau}_0\textstyle\int_1^0 f[\xi, \tau_0 + \theta(\bar{\tau}_0 - \tau_0), v_\nu[\xi', \tau_0 + \theta(\bar{\tau}_0 - \tau_0), \bar{\tau}_0, \bar{u}_0]d\theta \\ &+ \textstyle\int_{\tau_0}^{\tau} a[\xi, \tau', v_\nu, \tau', \bar{v}_\nu; 0, b_\nu[\xi', \tau', \tau_0, u_0, \tau'\bar{\tau}_0, \bar{u}_0; 0, \tilde{\tau}_0, \tilde{u}_0]d\tau' \end{aligned} \tag{7}$$

where v_ν and $\bar{v}_\nu$ stand for $v_\nu[\xi, \tau, \tau_0, u_0]$ and $v_\nu[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$ respectively.

LEMMA 3. *The difference functions b_ν of the approximating functionals have a common constant λ associated with them.*

In the first place it is clear from the definition of b_0 that

$$|b_0| \leqq |||\bar{\tau}|, |\bar{\tau}_0|, |\bar{u}_0|||.$$

Furthermore it follows from (7) and the properties of f and a that if there exist functions $b_\nu(\tau)$ such that

$$|b_\nu| \leqq \beta_\nu(\tau)\{|||\bar{\tau}|, |\bar{\tau}_0|, |\bar{u}_0|||\},$$

then

$$|b_{\nu+1}| \leqq \{1 + 2\gamma + |\textstyle\int_{\tau_0}^{\tau} \kappa\beta_\nu(\tau')d\tau'|\}\{|||\bar{\tau}|, |\bar{\tau}_0|, |\bar{u}_0|||\},$$

so that

$$\beta_{\nu+1}(\tau) = 1 + 2\gamma + |\textstyle\int_{\tau_0}^{\tau} \kappa\beta_\nu(\tau')d\tau'|.$$

One obtains therefore the following expressions of the functions $\beta_\nu(\tau)$

$$\beta_0 = 1,$$

$$\beta_1 = (1 + 2\gamma) + \frac{\kappa|\tau - \tau_0|}{1!},$$

$$\beta_2 = (1 + 2\gamma) + \frac{(1 + 2\gamma)^{\kappa|\tau - \tau_0|}}{1!} + \frac{\kappa^2|\tau - \tau_0|^2}{2!} \leqq (1 + 2\gamma)e^{\kappa\rho},$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$\beta_{\nu+1} = (1 + 2\gamma)\left\{1 + \frac{\kappa|\tau - \tau_0|}{1!} + \cdots \frac{\kappa^\nu|\tau - \tau_0|^\nu}{\nu!}\right\} + \frac{\kappa^{\nu+1}|\tau - \tau_0|^{\nu+1}}{(\nu + 1)!} \leqq (1 + 2\gamma)e^{\kappa\rho}.$$

Hence the common constant λ is given by $(1 + 2\gamma)e^{\kappa\rho}$ where $\rho = |\tau - \tau_0|$.

LEMMA 4. *If $p_\nu[\xi, \tau, \tau_0, u_0]$ and $p[\xi, \tau, \tau_0, u_0]$ are continuous in (B_0) and $\lim p_\nu = p$ uniformly in (B_0), then $\lim c_\nu(p_\nu) = c(p)$, where*

$$c_\nu[p_\nu] = \textstyle\int_{\tau_0}^{\tau} a[\xi, \tau', v_\nu, \tau', \bar{v}_\nu; 0, p_\nu[\xi, \tau, \tau_0, u_0]]d\tau',$$
$$c[p] = \textstyle\int_{\tau_0}^{\tau} a[\xi, \tau', v, \tau', \bar{v}; 0, p[\xi, \tau, \tau_0, u_0]]d\tau'.$$

It is seen that $c[p]$ is obtained from $c_\nu[p_\nu]$ by substituting for v_ν and $\bar{v}_\nu$ the limits v and $\bar{v}$ respectively as $\nu \to \infty$, and putting for p_ν the limit p. One has from the linearity property of

$$c[p] - c_\nu[p_\nu] = \textstyle\int_{\tau_0}^{\tau} \{a[\xi, \tau', v_\nu, \tau', \bar{v}_\nu; 0, p] - a[\xi, \tau', v, \tau', \bar{v}; 0, p]\}d\tau' + \textstyle\int_{\tau_0}^{\tau} a[\xi, \tau', v_\nu, \tau', \bar{v}_\nu; 0, p_\nu - p].$$

The integrand of the second term on the right does not exceed $\mu \,||\, p_\nu - p \,||$ so that it can be made as small as desired because of the uniformity of the

approach of p_ν to p. In the first term regard v_ν and $\bar{v}_\nu$ as depending upon ν', ξ, τ, τ_0, u_0 and ν', ξ, $\bar{\tau}$, $\bar{\tau}_0, u_0$, where $\nu' = 1/\nu$. Since $\lim\limits_{\nu \to \infty} v_\nu[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] = v[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$ and $\lim\limits_{\nu \to \infty} v_\nu[\xi, \tau, \tau_0, u_0] = v[\xi, \tau, \tau_0, u_0]$ uniformly in (B_0), it follows that $v[\nu', \xi, \tau, \tau_0, u_0]$ and $v[\nu', \xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$ are continuous functions of their arguments in the sense of section 2 so that property (1′) of a difference function is applicable, thus showing that $\lim\limits_{\nu \to \infty} c_\nu[p_\nu] = c[p]$ as desired.

LEMMA 5. *The sequence* $\{b_\nu\}$ *converges uniformly in* (B_0) *to a limit* b *given by the expression* (11) *below.*

By Lemma 2 of this section one may write

$$(8) \qquad b_{\nu+1} = d_\nu + c_\nu[b_\nu],$$

where

$$(9) \quad d_\nu = \tilde{u}_0(\xi) + \tilde{\tau} \int_0^1 f[\xi, \tau + \theta(\bar{\tau} - \tau), v_\nu[\xi', \tau + \theta(\bar{\tau} - \tau), \bar{\tau}_0, \bar{u}_0]] d\theta;$$
$$+ \tilde{\tau}_0 \int_1^0 f[\xi, \tau_0 + \theta(\bar{\tau}_0 - \tau_0), v_\nu[\xi', \tau_0 + \theta(\bar{\tau}_0 - \tau_0), \bar{\tau}_0, \bar{u}_0]] d\theta$$

and

$$(10) \quad c_\nu[b_\nu] = \int_{\tau_0}^{\tau} a[\xi, \tau', v_\nu, \tau', \bar{v}_\nu; 0, b_\nu[\xi', \tau', \tau_0, u_0, \tau', \bar{\tau}_0 \bar{u}_0; 0, \tilde{\tau}_0, \bar{u}_0]] d\tau'.$$

Repeating (8) ν times one obtains

$$b_{\nu+1} = d_\nu + c_\nu[d_{\nu-1}] + c_\nu c_{\nu-1}[d_{\nu-2}] + \cdots + c_\nu c_{\nu-1} \cdots c_1[b_0],$$

where the notation explains itself.

Consider also the infinite sum

$$(11) \qquad b = d + c[d] + c^2[d] + \cdots + c^\kappa[d] + \cdots,$$

where d and $c[d]$ have precisely the form of d_ν and $c_\nu[d_\nu]$ only with v, $\bar{v}$ substituted for v_ν and $\bar{v}_\nu$ respectively, and where c^κ means that the operation c is repeated κ times. It has already been seen that each term of (10) is dominated by the corresponding term of $(1 + 2\gamma)e^{\kappa\rho} \| |\tilde{\tau}|, |\tilde{\tau}_0|, |u_0| \|$. In a similar way it can be shown that the series for the last expression also dominates term by term the series (11). Hence there exists an integer κ_0 of such a character that all the terms of $b_{\nu+1}$ after the κ_0th have a sum which is in absolute value at most $\epsilon/3$ where ϵ is an arbitrarily assigned positive number. The same is true for the series representing b. Moreover, by repeated application of Lemma 4, one sees that there exists an index ν_0 such that if $\nu > \nu_0$, the first κ_0 terms of $b_{\nu+1}$ can be made to differ from the first κ_0 terms of b by a number which in absolute value is at most $\epsilon/3$. Hence $\lim b_\nu = b$ uniformly in (B_0).

Recalling now that the ν_ν converge to the solution v one sees that the hypothesis $H_{(1)}$ of Lemma 4, section 2 is satisfied. Furthermore, hypothesis $H_{(2)}$ of that Lemma is contained in Lemmas 2 and 3 of this section and $H_{(3)}$ is nothing but Lemma 5. Thus the conclusions of Lemma 4 are applicable

so that the solution $v[\xi, \tau, \tau_0, u_0]$ has a difference function b in (B_0) as desired.

§ 5. DIFFERENTIAL EQUATIONS INVOLVING SCHMIDT INTEGRAL POWER SERIES.

Let $u(\xi)$ be a real continuous function on the range (01). Then the function of $r + 1$ arguments

$$u^p(\xi)u^{p_1}(\xi_1)\cdots u^{p_r}(\xi_r) \tag{12}$$

where $\xi_1 \cdots \xi_r$ also range over (01) and $p, p_1, \cdots p_r$ are equal or distinct integers is evidently continuous. If now (12) is multiplied by an arbitrary function $\kappa(\xi, \xi_1, \cdots \xi_r)$ continuous in its arguments and the whole expression is integrated with respect to $\xi_1 \cdots \xi_r$, then the resulting function

$$w[\xi, u] = \int_0^1\cdots\int_0^1\kappa(\xi, \xi_1, \cdots \xi_r)u^p(\xi)u^{p_1}(\xi_1)\cdots u^{p_r}(\xi_r)d\xi_1\cdots d\xi_r{}^* \tag{13}$$

is again a continuous function of ξ. The expression

$$\mathfrak{A}[\xi, u] = w_1[\xi, u] + w_2[\xi, u] + \cdots + w_n[\xi, u]$$

is called a homogeneous integral power form of the mth order (Schmidt) if $w_1 \cdots w_n$ all have the form (13), and if for each $w_i, p + p_1 + \cdots + p_r = m$, where we may suppose always $m \leqq p$. For example, an integral power form of the 2d order has the following form,

$$\int_0^1\cdots\int_0^1\{\kappa_{200}u^2(\xi) + \kappa_{110}u(\xi)u(\xi_1) + \kappa_{020}u^2(\xi_1) + \kappa_{011}u(\xi_1)u(\xi_2)\}d\xi_1\cdots d\xi_r,$$

when all the κ's are arbitrary continuous functions of the $r + 1$ arguments $\xi, \xi_1, \cdots \xi_r$. This evidently reduces to

$$\alpha(\xi)u^2(\xi) + u(\xi)\int_0^1\beta(\xi, \xi_1)u(\xi_1)d\xi_1 + \int_0^1\gamma(\xi, \xi_1)u^2(\xi_1)d\xi_1 + \int_0^1\int_0^1\epsilon(\xi, \xi_1, \xi_2)u(\xi_1)u(\xi_2)d\xi_1 d\xi_2.$$

The κ's or the $\alpha, \beta, \gamma \cdots$ are called the coefficients by the form.

Let $\mathfrak{A}[\xi, u]$ be an integral power form of the mth order and let u be replaced by $u + \tilde{u}$, where $\tilde{u}$ is a continuous function of ξ. Then the sum of the terms involving the first power of $\tilde{u}$ is called the *first differential of* $\mathfrak{A}[\xi, u]$ and is denoted by $\mathfrak{A}[\xi, u, \tilde{u}]$ (Kowalewski). It is clear that when u is fixed $\mathfrak{A}[\xi, u, \tilde{u}]$ is an integral power form of the 1st order in $\tilde{u}$ and for $\tilde{u}$ fixed, of the $(m - 1)$st order in u. This differential is precisely the difference function defined in section 2 when $\bar{u} = u$. For example to compute the difference function of an integral power form of the 2d order one considers the expression

$$\begin{aligned}\mathfrak{A}[\xi, \bar{u}] - \mathfrak{A}[\xi, u] = \alpha(\xi)[\bar{u}^2(\xi) - u^2(\xi)] + \bar{u}(\xi)\int_0^1\beta(\xi, \xi_1)\bar{u}(\xi_1)d\xi_1 \\ - u(\xi)\int_0^1\beta(\xi, \xi_1)u(\xi_1)d\xi_1 + \int_0^1\gamma(\xi, \xi_1)[\bar{u}^2(\xi_1) - u^2(\xi_1)]d\xi_1 \\ + \int_0^1\int_0^1\epsilon(\xi, \xi_1, \xi_2)[\bar{u}(\xi_1)\bar{u}(\xi_2) - u(\xi_1)u(\xi_2)]d\xi_1 d\xi_2,\end{aligned}$$

* It is clear that the variables of integration $\xi_1,\cdots\xi_r$ may be so arranged that $p_1 \geqq p_2 \geqq \cdots \geqq p_r$ and this will be assumed throughout the section. For an account of this work see Kowalewski, loc. cit.

so that the difference function is

$$a[\xi, u, \bar{u}; \tilde{u}] = \alpha(\xi)[\bar{u}(\xi) + u(\xi)]\tilde{u}(\xi) + \tilde{u}(\xi)\int_0^1\beta(\xi, \xi_1)\bar{u}(\xi_1)d\xi_1$$
$$+ u(\xi)\int_0^1 \beta(\xi, \xi_1)\tilde{u}(\xi_1)d\xi_1 + \int_0^1\gamma(\xi, \xi_1)[\bar{u}(\xi_1) + u(\xi_1)]\tilde{u}(\xi_1)d\xi_1$$
$$+ \int_0^1\int_0^1\epsilon(\xi, \xi_1, \xi_2)[\tilde{u}(\xi_1)\bar{u}(\xi_2) + u(\xi_1)\tilde{u}(\xi_2)]d\xi_1 d\xi_2.$$

The differential is obtained by putting $\bar{u} = u$ and is

$$\mathfrak{A}[\xi, u, u; \tilde{u}] = 2\alpha(\xi)u(\xi)\tilde{u}(\xi) + \tilde{u}(\xi)\int_0^1\beta(\xi, \xi_1)u(\xi_1)d\xi_1$$
$$+ u(\xi)\int_0^1\beta(\xi, \xi_1)\tilde{u}(\xi_1)d\xi_1 + 2\int_0^1\gamma(\xi, \xi_1)u(\xi_1)\tilde{u}(\xi_1)d\xi_1$$
$$+ 2\int_0^1\int_0^1\epsilon(\xi, \xi_1, \xi_2)\tilde{u}(\xi_1)u(\xi_2)d\xi_1 d\xi_2.$$

Consider now the first differential $\mathfrak{A}[\xi, u, \tilde{u}]$ of the integral power form $\mathfrak{A}[\xi, u]$ of the mth order. If one puts in this $u + \hat{u}$ for u and arranges the result according to the powers of $\hat{u}$, then the sum of the terms of the first power in $\hat{u}$ is called the second differential of $\mathfrak{A}[\xi, u]$ and is denoted by $\mathfrak{A}[\xi, u; \tilde{u}, \hat{u}]$. In a similar way one could define the higher differentials of $\mathfrak{A}[\xi, u]$. One may easily deduce the formula

$$\text{(14)} \qquad \mathfrak{A}[\xi, u + \tilde{u}] = \mathfrak{A}[\xi, u] + \frac{\mathfrak{A}[\xi, u; \tilde{u}]}{1!} + \frac{\mathfrak{A}[\xi, u; \tilde{u}, \tilde{u}]}{2!} + \cdots$$

with the analogues of Euler's formulas

$$\mathfrak{A}[\xi, u; u] = m\mathfrak{A}[\xi, u], \qquad \mathfrak{A}[\xi, u; u, u] = m(m - 1)\mathfrak{A}[\xi, u], \cdots$$

which may be obtained from the obvious relation

$$\mathfrak{A}[\xi, cu] = c^m\mathfrak{A}[\xi, u],$$

where c is an arbitrary constant.

If in $\mathfrak{A}[\xi, u]$ all the coefficients are replaced by their absolute values and $u(\xi)$ by 1, one obtains a function of ξ whose maximum is called the height (Höhe) of $\mathfrak{A}[\xi, u]$. In a similar way one may define the height of $\mathfrak{A}[\xi, u; \tilde{u}]$, $\mathfrak{A}[\xi, u; \tilde{u}, \hat{u}]$, etc., by replacing the coefficients by their absolute values and $u, \tilde{u}, \hat{u}, \cdots$ by 1 and then taking the maximum with respect to ξ. If μ is the height of $\mathfrak{A}[\xi, \mu]$, then by Euler's relations $m\mu$ is the height of $\mathfrak{A}[\xi, u; \tilde{u}]$, $m(m - 1)\mu$ is the height of $\mathfrak{A}[\xi, u; \tilde{u}, \hat{u}]$ etc.

One may now with Schmidt consider infinite sums of integral power forms (integral power series), viz.,

$$\mathfrak{P}[\xi, u] = \mathfrak{A}_0[\xi, u] + \mathfrak{A}_1[\xi, u] + \mathfrak{A}_2[\xi, u] + \cdots,$$

where $\mathfrak{A}_p[\xi, u]$ is an integral power form of the pth order. Let μ_p represent the heights of $\mathfrak{A}_p[\xi, u]$ and consider the series

$$\mu_0 + \mu_1 x + \mu_2 x^2 + \cdots.$$

If the radius of convergence R of this power series is different from zero,

then the integral power series is said to be *regular* for $|u(\xi)| \leqq R$. If $|u(\xi)| \leqq r < R$, then the above power series is dominated by

$$P(r) = \mu_0 + \mu_1 r + \mu_2 r^2 + \cdots$$

since $|\mathfrak{A}_p[\xi, u]| \leqq \mu_p r^p$. If $|u(\xi)| \leqq r$ and $|\tilde{u}(\xi)| \leqq s$, $r + s < R$, one has immediately the inequalities

$$\begin{aligned} &|\mathfrak{A}_p[\xi, u]| \leqq \mu_p r^p, \qquad |\mathfrak{A}_p[\xi, u; \tilde{u}]| \leqq p\mu_p r^{p-1} s, \\ &\qquad |\mathfrak{A}_p[\xi, u; \tilde{u}, \tilde{u}]| \leqq p(p-1)\mu_p r^{p-2} s^2. \end{aligned} \tag{15}$$

Moreover Kowalewski shows by means of these that the following expansion is valid:

$$\mathfrak{P}[\xi, u + \tilde{u}] = \mathfrak{P}[\xi, u] + \mathfrak{P}[\xi, u; \tilde{u}] + \tfrac{1}{2}\mathfrak{P}[\xi, u; \tilde{u}, \tilde{u}] + \cdots, \tag{16}$$

where

$$\begin{aligned} \mathfrak{P}[\xi, u; \tilde{u}] &= \mathfrak{A}_1[\xi, u; \tilde{u}] + \mathfrak{A}_2[\xi, u; \tilde{u}] + \cdots, \\ \mathfrak{P}[\xi, u; \tilde{u}, \tilde{u}] &= \mathfrak{A}_2[\xi, u; \tilde{u}, \tilde{u}] + \mathfrak{A}_3[\xi, u; \tilde{u}, \tilde{u}] + \cdots, \text{ etc.,} \end{aligned}$$

From the inequalities (15) one obtains directly the dominance relation

$$\sum_{p,q}^{0\infty} \frac{1}{p!} |\mathfrak{A}_{p+q}[\xi, u; \overset{1}{\tilde{u}}\, \overset{2}{\tilde{u}} \cdots \overset{p}{\tilde{u}}]| \ll \sum_{p,q}^{0\infty} \frac{(p+q)\cdots(q+1)}{p!} \mu_{p+q} r^q s^p, \tag{17}$$

which may also be written with the help of (16) supposing $r + s < R$,

$$|\mathfrak{P}[\xi, u + \tilde{u}]| \ll P(r + s). \tag{18}$$

Furthermore

$$\begin{aligned} &|\mathfrak{P}[\xi, u; \tilde{u}]| \ll P'(r) s, \\ &|\mathfrak{P}[\xi, u; \tilde{u}, \tilde{u}]| \ll P''(r) s^2, \qquad \text{etc.,} \end{aligned} \tag{18'}$$

where $P'(r)$, $P''(r)$, etc., are the first, second, etc., derived series of $P(r)$.

It is desired now to investigate the differential equation containing integral power series

$$\begin{cases} \dfrac{\partial u(\xi, \tau)}{\partial \tau} = \mathfrak{P}[\xi, u], \\ u(\xi, 0) = 0 \qquad (\xi) \end{cases} \tag{19}$$

and see if the theorems proved in sections 1–4 are applicable. Kowalewski proves that such equations have solutions by a method very much analogous to that employed in proving the existence theorem for differential equations involving analytic functions. He shows in fact that the solution of

$$\frac{\partial u(\xi, \tau)}{\partial \tau} = \mathfrak{P}[\xi, u]$$

may be developed as a power series in τ with regular integral power series in u for coefficients.

It will be shown that the theory of the system (19) may be considered as a special instance of the theory described in this paper.

The following theorem is true.

THEOREM I'. *If* $\mathfrak{P}[\xi, u]$ *is a regular integral power series for* $\|u\| \leqq R$, *then the system* (19) *has a unique solution* $v(\xi, \tau)$ *continuous in*

$$(B') \qquad 0 \leqq \xi \leqq 1, \quad |\tau| \leqq \rho, \qquad \left(\rho = \frac{\beta}{P(R)}, \quad \beta = \frac{R}{3}\right).$$

The set (A') is now given by

$$(A') \qquad 0 \leqq \xi \leqq 1, \qquad |\tau| < \infty, \qquad \|u\| \leqq \beta.$$

In the first place, since each term of the integral power series is a continuous function of ξ, and since the series is uniformly convergent with regard to ξ for each fixed u of (A'), it is clear that hypothesis (H_1) of Theorem I is satisfied. To prove (H_2), replace in (16) the variables $u, \bar{u}$ by $u, u' - u$ respectively, with u, u' both satisfying (A'). Then, for each such u and u', one has

$$(20) \quad \mathfrak{P}[\xi, u'] - \mathfrak{P}[\xi, u] = \mathfrak{P}[\xi, u; u' - u] + \tfrac{1}{2}\mathfrak{P}[\xi, u; u' - u, u' - u] + \cdots$$

so that by the inequalities (18'), one finds that the right-hand side of the last equation is dominated by

$$s\left[P'(\beta) + \frac{s}{2}P''(\beta) + \cdots\right]$$

where $s = \|u' - u\|$. Hence, since $s = \|u' - u\| \leqq 2\beta$ and $\gamma + s \leqq 3\beta < \kappa$, it follows from (17) and (18) that

$$|\mathfrak{P}[\xi, u'] - \mathfrak{P}[\xi, u]| \leqq sP'(\beta + s) \leqq P'(R)\,\|u' - u\|.$$

Thus the hypotheses of Theorem I are satisfied. From the statements just preceding equation (15), it is clear that the value $\gamma = P(R)$ is an upper bound for $\mathfrak{P}[\xi, u]$.

The theorem on the continuity with regard to the initial elements reads precisely as Theorem III and will not be stated here.

THEOREM IV'. *If* $\mathfrak{P}[\xi, u]$ *is a regular integral power series for* $\|u\| \leqq R$, *then in the set*

$$(B_0') \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \rho, \qquad |\tau_0| \leqq \delta, \qquad \|u_0\| \leqq \delta,$$

the solution $v[\xi, \tau, \tau_0, u_0]$ *has a difference function* $b[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \tilde{\tau}, \tilde{\tau}_0\tilde{u}_0]$

The set corresponding to (A_0) is here given by

$$(A_0') \quad 0 \leqq \xi \leqq 1, \qquad -\infty < \tau < \infty, \qquad \|u\| \leqq \beta + \delta, \qquad (\beta + \delta < R/3).$$

It is requisite to show that the hypotheses of Theorem IV are satisfied. The proof that the hypotheses of Theorem I are satisfied in (A_0') is the same as that given in Theorem I$'$. To show that $\mathfrak{P}[\xi, u]$ has a difference function one makes use of the expansion (20). Consider the right-hand side with the last argument $u' - u$ of each term replaced by $\tilde{u}$,

$$a[\xi, u, u'; \tilde{u}] = \mathfrak{P}[\xi, u; \tilde{u}] + \tfrac{1}{2}\mathfrak{P}[\xi, u; u' - u, \tilde{u}] + \cdots.$$

The linearity of the functional a in $\tilde{u}$ is clear from the definitions of $\mathfrak{P}[\xi, u; \tilde{u}]$, $\mathfrak{P}[\xi, u; u' - u, \tilde{u}]$, etc. Furthermore, the functional a is continuous for every continuous function $\tilde{u}$. For, set $s = || u' - u ||$ and $S = || \tilde{u} ||$; then, by inequalities analogous to (15), (17) and (18$'$), it follows that

$$|\mathfrak{P}[\xi, u; \tilde{u}]| \ll P'(\beta)S \ll P'(R)S,$$
$$|\mathfrak{P}[\xi, u; u' - u, \tilde{u}]| \ll P''(\beta)Ss \ll P''(R)Ss, \text{ etc.},$$

Hence, the series for $a[\xi, u, u'; \tilde{u}]$ is dominated by $P'(\beta + s)S$ or $P'(R)S$; i.e., for a fixed $\tilde{u}$ it converges uniformly with respect to its other arguments and consequently represents a continuous functional for each fixed $\tilde{u}$. The constant μ associated with the difference function is $P'(R)$.

It remains to prove that a has the property (1$'$). In the first place it can readily be shown by a method of proof similar to that used in the proof of Lemma 1, section 2, that, if each of a sequence of functionals has the property (1$'$), and the sequence converges uniformly in the set under consideration, then the limit functional also has the property (1$'$). Since the above series for a does indeed converge uniformly, it suffices to prove that the difference function of (13) has the property (1$'$). But it follows from the definition of a difference function and from the form of (13) that the difference function of (13) is a sum of integrals of the same type except that in the integrand product there occur two functions u, $\tilde{u}$ instead of a single function u. That integrals of this last type have property (1$'$), follows very readily from the fact that the integrals are continuous functionals of u and $\tilde{u}$.